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ON THE SOLVABILITY OF THE NONLINEAR PROBLEM FOR ELLIPTIC-PARABOLIC SYSTEM OF THE EQUATIONS IN HÖLDER SPACES

There is proved the existence and the uniqueness of the solution of the nonlinear H.Amann problem for the parabolic-elliptic equations for the small time in the Holder spaces, the estimates for the solution are derived, the smoothness on t of the potential φ is obtained.

1. The statement of the problem.

Let $\Omega \in R^n$ be bounded domain, $S = \partial\Omega$, $\Omega_T = \Omega\times(0,T)$, $S_T = S\times[0,T]$. Let $u(x,t) = (u_1,\dots,u_N)$, $m(x,t,u) = (m_1,\dots,m_N)$, $f(x,t,u,\partial_x u) = (f_1,\dots,f_N)$, $g(x,t,u,\varphi) = (g_1,\dots,g_N)$ be vector -functions; u^T = colon$(u_1,\dots,u_N)$- column-vector; $z = (z_1,\dots,z_N) \in R^N$, $z^2 := z_1^2 + \dots + z_N^2 \neq 0$; d_k, $c_k, k = 1,\dots$– positive constants; $\alpha \in (0,1)$; $a\,b^T := (a_1,\dots,a_N)\,(b_1,\dots,b_N)^T = \sum_{r=1}^N a_r\,b_r$, $a^T\,b = \{a_r\,b_s\}_{1\le r,s\le N}$; E–identity $N\times N$ matrix; $A(x,t,u) = \{a_{rs}\}_{1\le r,s\le N}$–the matrix with coefficients satisfying the conditions

$$\sum_{r,s=1}^N a_{rs}(x,t,u)\xi_r\xi_s \ge \mu_0\xi^2,\ a_{rs} = a_{sr},\ r,s = 1,\dots,N,\ \mu_0 = \text{const} > 0. \tag{1}$$

We consider the problem with unknown functions $u(x,t)$ and $\varphi(x,t)$

$$\partial_t u^T - A(x,t,u)\,\Delta u^T - m^T(x,t,u)\Delta\varphi = f^T(x,t,u,\partial_x u) \text{ in } \Omega_T, \tag{2}$$

$$zu^T := z_1u_1 + \dots + z_Nu_N = 0 \text{ in } \bar\Omega_T, \tag{3}$$

$$u_r|_{t=0} = u_{0r}(x),\ r = 1,\dots,N, \text{ in } \Omega, \tag{4}$$

$$A(x,t,u)\partial_\nu u^T + m^T(x,t,u)\,\partial_\nu\varphi = g^T(x,t,u,\varphi) \text{ on } S_T, \tag{5}$$

where ν is the outer normal to S. There is assumed that $N\ge 2$, otherwise if $N = 1$ we obtain from the equation (3) $u_1 = 0$ and the problem (2)–(5) is reduced to the elliptic problem with unknown function $\varphi(x,t)$. In this problem there are $N+1$ unknown functions and N boundary conditions (5), but by virtue of the condition $z^2\neq 0$ one of the unknown functions $u_1,\dots,u_N$ is found from the equation (3).

This problem was set up by H.Amann [1]. It describes the process of the change of concentrations $u_1,\dots,u_N$ of each of N substances with charges $z_1,\dots,z_N$ in the solute under the action of the electric field with potential φ. H.Amann and M.Renardy [2] proved the existence and uniqueness of the solution to the problem $u_r \in C(\bar\Omega\times[0,t^+))\cap C^{1,0}(\bar\Omega\times(0,t^+))\cap C^{2,1}(\Omega\times(0,t^+))$, $r = 1,\dots,N$, $\varphi\in C^{2,0}(\bar\Omega\times(0,t^+))$. S.I.Pokhozhaev has studied the problem for the elliptic equations in Sobolev spaces [3]. We note also that H.Amann problem is not embedded into the general theory of the boundary value problems for parabolic and elliptic equations.

In the present paper the problem (2)–(5) is studied in the Holder spaces $C_{x\ t}^{l,l/2}(\bar\Omega_T)$ for $t\le T_0$ [4], the smoothness of potential φ with respect to t is determined.

At first, we find the function $\varphi(x,0) := \varphi_0(x)$. For that we multiply from the left both parts of the equation (2) and boundary condition (5) by the vector z, in the equation we take into account the identity $z\partial_t u^T = 0$, which follows from the equation (3), and put $t = 0$ in the obtained equation and boundary condition, then we derive the elliptic problem for the function $\varphi_0(x)$

$$\Delta \varphi_0 = f_0(x) \text{ in } \Omega, \tag{6}$$

$$\partial_\nu \varphi_0 - G_0(x, \varphi_0) = p_0(x) \text{ on } S, \tag{7}$$

here

$$f_0 = -\frac{1}{\kappa_0(x)}\big(zA(x, 0, u_0(x))\Delta u_0^T(x) + zf^T(x, 0, u_0(x), \partial_x u_0(x)\big),$$

$$p_0(x) = -\frac{1}{\kappa_0(x)} zA(x, 0, u_0(x))\partial_\nu u_0^T(x), G_0 = \frac{1}{\kappa_0(x)}\, z\, g^T(x, 0, u_0(x), \varphi_0(x)),$$

$$\kappa_0(x) = zm^T(x, 0, u_0(x)) \neq 0 \text{ in } \bar{\Omega}.$$

Lemma 1. *Let* $S \in C^{2+\alpha}, \alpha \in (0, 1),\ G_0(x, q) \in C^3(R^1; C^{1+\alpha}(S))$. *We assume*

$$|\kappa_0| \geq d_1\ \forall\, x \in \bar{\Omega},\ \partial_q G_0(x, q) \leq -d_2\ \forall x \in S,\ q \in R^1.$$

Then for every functions $f_0 \in C^\alpha(\bar{\Omega})$, $p_0 \in C^{1+\alpha}(S)$ *the problem (6), (7) has unique solution* $\varphi_0(x) \in C^{2+\alpha}(\bar{\Omega})$ *and the estimate for it is valid*

$$|\varphi_0|_\Omega^{(2+\alpha)} \leq c_1\big(|f_0|_\Omega^{(\alpha)} + |p_0|_S^{(1+\alpha)}\big).$$

We denote

$$K_N = \{l = (l_1, \ldots, l_N) \mid d_3 < |l_r| < d_4,\ l_r \in R^1,\ r = 1, \ldots, N\},$$

$$K_{N,n} = \{\mathcal{C} = \{c_{ri}\}_{\substack{1 \leq r \leq N \\ 1 \leq i \leq n}} \mid |c_{ri}| < d_5,\ c_{ri} \in R^1\},\ \kappa(x, t, l) = \sum_{r=1}^{N} z_r\, m_r(x, t, l),$$

$$æ = z\, A^{-1}(x, t, l)(E - m^T(x, t, l)\, z/\kappa(x, t, l))\, \partial_q\, g^T(x, t, l, q).$$

We shall consider the problem (2)–(5) under the following assumptions:

A) $a_{rs}(x, t, p), m_r(x, t, p) \in C^3(\bar{K}_N;\ C^{1+\alpha,(1+\alpha)/2}_{x\ t}(\bar{\Omega}_T))$; $f_r(x, t, l, \mathcal{C}) \in C^2(\bar{K}_N \times \bar{K}_{N,n}$; $C^{\alpha,\alpha/2}_{x\ t}(\bar{\Omega}_T))$; $g_r(x, t, l, q) \in C^4(\bar{K}_N \times R^1; C^{1+\alpha,(1+\alpha)/2}_{x\ t}(\bar{\Omega}_T))$ and $z\, \partial_q g^T \leq -d_2\ \forall (x, t) \in S_T,\ l \in \bar{K}_N,\ q \in R^1,\ r, s = 1, \ldots, N$;

B) $z = (z_1, \ldots, z_N) \in R^N,\ z^2 = z_1^2 + \ldots + z_N^2 \neq 0$;

C) $|\kappa(x, t, l)| \geq d_6,\ |æ(x, t, l, q)| \geq d_6\ \forall\ (x, t, l) \in \bar{\Omega}_T \times \bar{K}_N, q \in R^1$ and $\partial_q\, g^T \neq c(x, t)m^T$, $c-$ arbitrary scalar function.

We formulate the main result of the present paper.

Теорема 1. *Let* $S \in C^{2+\alpha}$, $\alpha \in (0, 1)$, *and the conditions A)–C) be fulfilled. Then for arbitrary functions* $u_{0r}(x) \in C^{2+\alpha}(\bar{\Omega})$, $r = 1, \ldots, N$, *satisfying N compatibility conditions there exists* $T_0 \in (0, T]$ *such that*

1. *the problem (2)–(5) has unique solution* $u_r \in C^{2+\alpha,1+\alpha/2}_{x\ t}(\bar\Omega_{T_0})$, $\varphi_1 := \varphi(x,t) - \varphi_0(x) \in C^{2+\alpha}_x(\bar\Omega_{T_0})$, $\Delta\varphi_1 \in \overset{\circ}{C}{}^{\alpha,\alpha/2}_{x\ t}(\bar\Omega_{T_0})$, $\varphi_1|_S$, $\partial_\nu\varphi_1|_S \in \overset{\circ}{C}{}^{1+\alpha,(1+\alpha)/2}_{x\ t}(S_{T_0})$ *and the estimate takes place*

$$\sum_{r=1}^N |u_r|^{(2+\alpha)}_{\Omega_t} + |\varphi_1|^{(2+\alpha)}_{x,\Omega_t} + |\Delta\varphi_1|^{(\alpha)}_{\Omega_t} + |\varphi_1|^{(1+\alpha)}_{S_t} + |\partial_\nu\varphi_1|^{(1+\alpha)}_{S_t} \le$$

$$\le c_2\Big(\sum_{r=1}^N |u_{0r}|^{(2+\alpha)}_{\Omega} + |\varphi_0|^{(2+\alpha)}_{\Omega}\Big),\ t \le T_0, \tag{8}$$

where $\varphi_0 = \varphi(x,0)$ *is the solution of the problem (6), (7).*

2. *The function* φ_1 *satisfies the Holder condition with respect to* t *with index* $\alpha/2$ $\forall x \in \bar\Omega$. *For every compact set* $\bar\Omega' \subset \Omega$ *and* $\forall\beta \in (0,\alpha)$ *the derivatives* $\partial_x\varphi_1$ *and* $\partial_x^2\varphi_1$ *satisfy the Holder condition with respect to* t *with indexes* $\alpha/2$ *and* $(\alpha-\beta)/2$ $\forall x \in \bar\Omega'$ *respectively and the estimate for them is fulfilled*

$$[\varphi_1]^{(\alpha/2)}_{t,\Omega_t} + [\partial_x\varphi_1]^{(\alpha/2)}_{t,\Omega'_t} + [\partial_x^2\varphi_1]^{(\frac{\alpha-\beta}{2})}_{t,\Omega'_t} \le c_3\Big(\sum_{r=1}^N |u_{0r}|^{(2+\alpha)}_{\Omega} + |\varphi_0|^{(2+\alpha)}_{\Omega}\Big).$$

Here by $C^{2+\alpha}_x(\bar\Omega_T)$ we mean the set of functions $\varphi(x,t)$ with the norm [4]

$$|\varphi|^{(2+\alpha)}_{x,\Omega_T} = \sum_{|m|\le 2} |\partial_x^m\varphi|_{\Omega_T} + \sum_{|m|=2} [\partial_x^m\varphi]^{(\alpha)}_{x,\Omega_T}.$$

REMARK 1. *Theorem 1 is valid for the matrix* $A = E - m^T z/\kappa$ *with rank equaled to* $N-1$. *2. If* $\partial_q g^T = c(x,t)m^T$, $c-$ *arbitrary scalar function, then* $æ = c\,z\,A^{-1}(E - m^T z/\kappa)\,m^T = 0$, *as* $\kappa = z\,m^T$.

To reduce the problem (2)–(5) to the problem with zero initial data we construct auxiliary functions $V_r \in C^{2+\alpha,1+\alpha/2}_{x\ t}(R^n_T)$ under the conditions [4] $V_r|_{t=0} = \tilde u_{0r}(x)$, $\partial_t V_r|_{t=0} = \sum_{s=1}^N \tilde a_{rs}(x,0,\tilde u_0(x))\Delta\tilde u_{0s} + \tilde m_r(x,0,\tilde u_0)\Delta\tilde\varphi_0 + \tilde f_r(x,0,\tilde u_0(x),\partial_x\tilde u_0(x))$, here $R^n_T = R^n \times (0,T)$, the symbol $\sim$ means the continuation of the function into the entire space R^n with preserving of class. These functions obey the estimate

$$|V_r|^{(2+\alpha)}_{R^n_T} \le c_4\Big(\sum_{r=1}^N |u_{0r}|^{(2+\alpha)}_{\Omega} + |\varphi_0|^{(2+\alpha)}_{\Omega}\Big),\ r = 1,\dots,N. \tag{9}$$

In the problem (2)–(5) we make the substitution

$$u_r = v_r(x,t) + V_r(x,t),\ r = 1,\dots,N,\ \varphi(x,t) = \varphi_1(x,t) + \varphi_0(x), \tag{10}$$

and obtain the problem with unknown functions $v = (v_1,\dots,v_N)$, φ_1 satisfying zero initial data $\partial_t^k v_r|_{t=0} = 0$, $k = 0,1$, $\varphi_1|_{t=0} = 0$, in the form

$$\partial_t v^T - A(x,t,V)\Delta v^T - m^T(x,t,V)\Delta\varphi_1 =$$

$$= p^T(x,t) + P^T(x,t,v,\partial_x v,\Delta v,\Delta\varphi_1) \text{ in } \Omega_T, \tag{11}$$

$$zv^T = F(x,t) \text{ in } \bar{\Omega}_T, \tag{12}$$

$$A(x,t,V)\partial_\nu v^T + m^T(x,t,V)\,\partial_\nu\varphi_1 - G^T(x,t)\varphi_1 =$$
$$= q^T(x,t) + Q^T(x,t,v,\partial_\nu v,\varphi_1,\partial_\nu\varphi_1) \text{ on } S_T, \tag{13}$$

where $V=(V_1,...,V_N), p=(p_1,...,p_N),\ P=(P_1,...,P_N),\ q=(q_1,...,q_N)$, $Q=(Q_1,...,Q_N),\ G=(\partial_{\varphi_0}g_1(x,t,V,\varphi_0)\big|_S,\ldots,\partial_{\varphi_0}g_N(x,t,V,\varphi_0)\big|_S)$,

$$p_r = -\partial_t V_r + \sum_{s=1}^N a_{rs}(x,t,V)\,\Delta\,V_s + m_r(x,t,V)\,\Delta\,\varphi_0 + f_r(x,t,V,\partial_x V);$$

$$P_r = \sum_{s=1}^N \big(a_{rs}(x,t,v+V) - a_{rs}(x,t,V)\big)\,\Delta\,(v_s+V_s) + \Big(m_r(x,t,v+V)-$$
$$-m_r(x,t,V)\Big)\,\Delta\,(\varphi_1+\varphi_0) + f_r(x,t,v+V,\partial_x v+\partial_x V) - f_r(x,t,V,\partial_x V);$$

$$F(x,t) = -\sum_{r=1}^N z_r\,V_r(x,t);$$

$$q_r(x,t) = g_r(x,t,V,\varphi_0) - \sum_{s=1}^N a_{rs}(x,t,V)\,\partial_\nu V_s - m_r(x,t,V)\,\partial_\nu\varphi_0\big|_S;$$

$$Q_r = -\sum_{s=1}^N \big(a_{rs}(x,t,v+V) - a_{rs}(x,t,V)\big)\,\partial_\nu\,(v_s+V_s) - \Big(m_r(x,t,v+V)-$$
$$-m_r(x,t,V)\Big)\,\partial_\nu(\varphi_1+\varphi_0) + g_r(x,t,v+V,\varphi_1+\varphi_0) - g_r(x,t,V,\varphi_1+\varphi_0)+$$
$$+\varphi_1\int_0^1 [\partial_{\varphi_0}g_r(x,t,V,\ \varphi_0+\lambda\varphi_1) - \partial_{\varphi_0}g_r(x,t,V,\ \varphi_0)]\,d\lambda.$$

Here p_r, F, q_r are known functions, $p_r \in \overset{\circ}{C}{}^{\alpha,\alpha/2}_{x\ t}(\bar{\Omega}_T)$, $F \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{x\ t}(\bar{\Omega}_T), q_r \in \overset{\circ}{C}{}^{1+\alpha,(1+\alpha)/2}_{x\ t}(S_T)$, $r=1,\ldots,N$ and the estimate for them is valid

$$\sum_{r=1}^N(|p_r|^{(\alpha)}_{\Omega_T} + |q_r|^{(1+\alpha)}_{S_T}) + |F|^{(2+\alpha)}_{\Omega_T} \le c_5(\sum_{r=1}^N(|u_{0r}|^{(2+\alpha)}_{\Omega} + |\varphi_0|^{(2+\alpha)}_{\Omega}. \tag{14}$$

2. The linear problem.

We consider the linear problem with unknown functions $w(x,t)=(v,\varphi_1)$ satisfying zero initial data

$$\partial_t v^T - A(x,t)\,\Delta v^T - m^T(x,t)\Delta\varphi_1 = p^T(x,t) \text{ in } \Omega_T, \tag{15}$$

$$zv^T := z_1v_1 + \ldots + z_Nv_N = F(x,t) \text{ in } \bar{\Omega}_T, \tag{16}$$

$$A(x,t)\partial_\nu v^T + m^T(x,t)\,\partial_\nu\varphi_1 - G^T\varphi_1 = g^T(x,t) \text{ on } S_T, \tag{17}$$

where the matrix A satisfies the condition (1), $z = (z_1, \dots, z_N) \in R^N$, $z^2 \neq 0$; $v^T =$ colon$(v_1, \dots, v_N)$–column-vector.

THEOREM 2. *Let $S \in C^{2+\alpha}$, $\alpha \in (0,1)$. We assume that a_{rs}, m_r, $G_r \in C_{x\ t}^{1+\alpha, \frac{1+\alpha}{2}}(\bar{\Omega}_T)$, $r, s = 1, \dots, N$; $|\kappa|, |æ| \geq d_7$ in $\bar{\Omega}_T$. Then for every functions $p_r \in \overset{\circ}{C}{}_{x\ t}^{\alpha, \alpha/2}(\bar{\Omega}_T)$; $F \in \overset{\circ}{C}{}_{x\ t}^{2+\alpha, 1+\alpha/2}(\bar{\Omega}_T)$; $q_r \in \overset{\circ}{C}{}_{x\ t}^{1+\alpha, \frac{1+\alpha}{2}}(S_T), r = 0, 1, \dots, N$,*

1. *the problem (15)-(17) has unique solution $v_r \in \overset{\circ}{C}{}_{x\ t}^{2+\alpha, 1+\alpha/2}(\bar{\Omega}_{T_1}), r = 1, \dots, N, \varphi \in C_x^{2+\alpha}(\bar{\Omega}_T)$, $\Delta\varphi_1 \in C_{x\ t}^{\alpha, \alpha/2}(\bar{\Omega}_T); \varphi_1|_S, \partial_\nu \varphi_1|_S \in C_{x\ t}^{1+\alpha, (1+\alpha)/2}(S_T)$ and*

$$||w||_T := \sum_{r=1}^{N} |v_r|_{\Omega_T}^{(2+\alpha)} + |\varphi_1|_{x, \Omega_T}^{(2+\alpha)} \leq$$

$$\leq c_6 \Big(\sum_{r=1}^{N} (|p_r|_{\Omega_T}^{(\alpha)} + |q_r|_{S_T}^{(1+\alpha)}) + |F|_{\Omega_T}^{(2+\alpha)} \Big) := c_6 |||h|||_T, \tag{18}$$

$$|\Delta\varphi_1|_{\Omega_T}^{(\alpha)} + |\varphi_1|_{S_T}^{(1+\alpha)} + |\partial_\nu \varphi_1|_{S_T}^{(1+\alpha)} \leq c_7 |||h|||_T. \tag{19}$$

2. *The function φ_1 satisfies the Holder condition with respect on t with index $\alpha/2$ $\forall x \in \bar{\Omega}$. For any compact set $\bar{\Omega}' \in \Omega$ and $\forall \beta \in (0, \alpha)$ the derivatives $\partial_x \varphi_1$ and $\partial_x^2 \varphi_1$ satisfy the Holder conditions with respect on t with indexes $\alpha/2$ and $(\alpha - \beta)/2$ $\forall x \in \bar{\Omega}'$ respectively and the estimate is fulfilled*

$$[\varphi_1]_{t, \Omega_t}^{(\alpha/2)} + [\partial_x \varphi_1]_{t, \Omega_t'}^{(\alpha/2)} + [\partial_x^2 \varphi_1]_{t, \Omega_t'}^{(\frac{\alpha-\beta}{2})} \leq c_8 |||h|||_t, \ t \leq T. \tag{20}$$

Proof. 1. We reduce the problem (15)–(17) to the equivalent one. We multiply from the left both parts of the equation (15) by the vector z and boundary condition (17) by the vectors z and $z\,A^{-1}$, take into account that $z \partial_t u^T = \partial_t F$, $z\, \partial_\nu\, u^T = \partial_\nu F$, then we obtain

$$\Delta\varphi_1 = -\frac{1}{\kappa}(zA\Delta v^T - \partial_t F + z\, p^T) \text{ in } \Omega_T, \tag{21}$$

$$\partial_\nu \varphi_1 = \frac{1}{\kappa}(z\, G^T\, \varphi - z\, A \partial_\nu u^T + z\, q^T) \text{ on } S_T, \tag{22}$$

$$z\, A^{-1}\, m^T\, \partial_\nu \varphi_1 - z\, A^{-1}\, G^T\, \varphi_1 = z\, A^{-1}\, q^T - \partial_\nu F \text{ on } S_T. \tag{23}$$

We note, that the conditions (22), (23) are linear independent, because (22) is the linear combination of the boundary conditions (17), and the condition (23) is found with the help of the equation (16). Solving this equations we find

$$\varphi_1 = \frac{1}{æ}\Big(- z\, A^{-1} \frac{m^T\, z}{\kappa} A \partial_\nu v^T - z A^{-1} (E - \frac{m^T\, z}{\kappa}) q^T + \partial_\nu F \Big) \text{ on } S_T, \tag{24}$$

$$\partial_\nu \varphi_1 = \frac{z}{\kappa æ}\big((A^{-1} G^T z - G^T z A^{-1}) q^T + G^T \partial_\nu F - A^{-1} G^T z A \partial_\nu v^T \big) \text{ on } S_T, \tag{25}$$

where E – $N \times N$ identity matrix, $\kappa = z\, m^T$, $\text{æ} = z\, A^{-1}(E - m^T z/\kappa)$. Substituting the expression $\Delta\varphi_1$ (21) into the equation (15), the derivative $\partial_\nu \varphi_1|_S$ (22) and then $\varphi_1|_S$ (24) into the boundary condition (17) we obtain the problem for the function $v(x,t) = (v_1, \dots, v_N)$

$$\partial_t v^T - \big(E - \frac{m^T(x,t)\, z}{\kappa(x,t)}\big)\, A(x,t)\Delta u^T =$$

$$= \frac{m^T(x,t)}{\kappa(x,t)}\, \partial_t F(x,t) + \big(E - \frac{m^T(x,t)\, z}{\kappa(x,t)}\big)\, p^T(x,t) \text{ in } \Omega_T, \tag{26}$$

$$z v^T = F(x,t) \text{ in } \bar{\Omega}_T, \tag{27}$$

$$\big(E - \frac{m^T(x,t)\, z}{\kappa(x,t)}\big)\big[E + \frac{1}{\text{æ}(x,t)}\, G^T(x,t) z A^{-1}(x,t) \frac{m^T(x,t) z}{\kappa(x,t)}\big]\, A(x,t)\partial_\nu u^T =$$

$$= \big(E - \frac{m^T(x,t) z}{\kappa(x,t)}\big)\, q^T(x,t) \text{ on } S_T, \tag{28}$$

and the Neumann problem for the potential φ_1

$$\Delta\varphi_1(x,t) = \frac{1}{\kappa}(-zA\Delta v^T + \partial_t F - z\, p^T) \text{ in } \Omega_T, \tag{29}$$

$$\varphi_1|_{t=0} = 0 \text{ in } \Omega, \tag{30}$$

$$\partial_\nu \varphi_1 = \frac{1}{\kappa \text{æ}} z\big((A^{-1}G^T z - G^T z A^{-1}) q^T + G^T F_\nu - A^{-1} G^T z A \partial_\nu v^T\big) \text{ on } S_T. \tag{31}$$

We consider the problem (26)–(28). Here the matrix $E - m^T z/\kappa$ is projector with rank equaled to $N-1$ and the system (28) consists of $N-1$ conditions. It is seen that the matrix A may be equaled to $E - m^T z/\kappa$.

The estimate (18) and existence of the solution are derived with the help of Schauder [4] and parameter continuation methods respectively. For the applications of these methods it is required to have the estimate of the solution of model problem in half-space $x_n > 0$ and the solvability of the problem (26)–(28) with constant coefficients. In Ch.4 we give the scheme of the proof of the solvability of last one, which will be valid also for the model problem in $x_n > 0$, because the geometry of the domain is not influenced on the proof.

We consider Neumann problem (29)–(31) taking into account that $v_r \in C^{2+\alpha, 1+\alpha/2}_{x\ t}(\bar{\Omega}_T)$. By virtue of the initial condition (30) this problem has unique solution $\varphi_1 \in C^{2+\alpha}_x(\bar{\Omega}_T)$ and it satisfies the estimate $\forall t \le T$

$$|\varphi_1|^{(2+\alpha)}_{x,\Omega_t} \le c_9\Big(\sum_{r=1}^{N} (|p_r|^{(\alpha)}_{\Omega_t} + |q_r|^{(1+\alpha)}_{S_t} + |v_r|^{(2+\alpha)}_{\Omega_t}) + |F|^{(2+\alpha)}_{\Omega_t}\Big) := c_9 I(t). \tag{32}$$

From the identities (29), (24), (25) we obtain

$$\Delta\varphi_1 \in C^{\alpha,\alpha/2}_{x\ t}(\bar{\Omega}_T);\ \varphi_1|_S,\ \partial_\nu\varphi_1|_S \in C^{1+\alpha,(1+\alpha)/2}_{x\ t}(S_T)$$

and

$$|\Delta\varphi_1|^{(\alpha)}_{\Omega_t} + |\varphi_1|^{(1+\alpha)}_{S_t} + |\partial_\nu\varphi_1|^{(1+\alpha)}_{S_t} \le c_{10} I(t),\ t \le T, \tag{33}$$

that is the functions $\Delta\varphi_1$ and $\varphi_1|_S$, $\partial_\nu\varphi_1|_S$ obey the Holder condition on t with indexes $\alpha/2$ and $(1+\alpha)/2$ respectively. Application of the estimate (18) for the functions v_r in (32), (33) leads to the inequality (19).

2. For every function $\varphi_1 \in C_x^{2+\alpha}(\bar\Omega_T)$ there takes place the integral representation formulas [5]

$$\varphi_1(x,t) = \int_S \left[\partial_\nu\varphi_1(\xi,t)\mathcal{E}_n(x-\xi) - \varphi_1(\xi,t)\partial_\nu\mathcal{E}_n(x-\xi)\right] d_\xi S-$$

$$-\int_\Omega \Delta\varphi_1(\xi,t)\mathcal{E}_n(x-\xi)\,d\xi := I_1(x,t) - I_2(x,t) \equiv V(x,t), x\in\Omega, \tag{34}$$

$$\varphi_1(x_0,t) = 2V(x_0,t),\ x_0\in S, \tag{35}$$

where $\mathcal{E}_n(x-\xi)$ is the fundamental solution of Laplace equation,

$$\mathcal{E}_n(x) = \frac{1}{(n-2)æ_n}\frac{1}{|x|^{n-2}},\ n\geq 3;\ \mathcal{E}_n(x) = \frac{1}{2\pi}\ln\frac{1}{|x|},\ n=2,$$

$æ_n$– the area of unit sphere in R^n, ν–the outer normal to S.

It is known that if the density φ_1 is continuous on x and S is the Lyapunov surface then the direct value of the potential $W(x_0,t) = \int_S \varphi_1(\xi,t)\partial_\nu\mathcal{E}_n(x-\xi)d_\xi S, x_0\in S$ exists and is continuous on S, moreover the jump formula is valid

$$W_i(x_0,t) := \lim_{x\in\Omega, x\to x_0} W(x,t) = -\frac{1}{2}\varphi_1(x_0,t) + W(x_0,t). \tag{36}$$

We direct $x\in\Omega$ to $x_0\in S$ in the formula (34) taking into account (36)

$$\varphi_{1i}(x_0,t) := \lim_{x\in\Omega, x\to x_0}\varphi_1(x,t) = \frac{1}{2}\varphi_1(x_0,t) + V(x_0,t).$$

From here and formula (35) we obtain $\varphi_{1i}(x_0,t) = \varphi_1(x_0,t)$.

We estimate the function $\varphi_1(x,t)$ (34)

$$|\varphi_1(x,t) - \varphi_1(x,t_1)| \leq c_{11}\,\gamma\,|t-t_1|^{\alpha/2}, x\in\Omega, \tag{37}$$

where $\gamma = [\varphi_1]_{t,S_T}^{((1+\alpha)/2)} + [\partial_\nu\varphi_1]_{t,S_T}^{((1+\alpha)/2)} + [\Delta\varphi_1]_{t,\Omega_T}^{(\alpha/2)}$, and for the direct value of the function $\varphi_1(x_0,t)$ (35) we have

$$|\varphi_1(x_0,t) - \varphi_1(x_0,t_1)| \leq 2c_{11}\,\gamma\,|t-t_1|^{\alpha/2}\ \forall x_0\in S.$$

This inequality and (37) lead to the estimate

$$|\varphi_1(x,t) - \varphi_1(x,t_1)| \leq 2c_{11}\,\gamma\,|t-t_1|^{\alpha/2}\ \forall x\in\bar\Omega.$$

In any compact set $\bar\Omega' \subset \Omega$ the surface potentials $I_1(x,t)$ in (34) and their any derivatives with respect to x satisfy Holder condition on t with index $(1+\alpha)/2$. The derivatives $\partial_x I_2(x,t)$ and $\partial_x^2 I_2(x,t)$ obey Holder conditions on t with indexes $\alpha/2$ and $(\alpha-\beta)/2\ \forall\beta\in(0,\alpha)$. Thus the estimate (20) is derived.•

3. Proof of Theorem 1.

We reduce the nonlinear problem (11)–(13) to the problems (26)–(28) and (29)–(31) with $p^T + P^T$ and $q^T + Q^T$ instead of p^T and q^T and write it down in the operator form

$$K\,w^T = h^T + N\,w^T, \tag{38}$$

where $w = (v_1, \ldots, v_N, \varphi)$; K–linear operator determined by the operators in the left-hand parts of the equations and conditions of the problems (26)–(28) and (29)–(31), $h = (p,\, F,\, q)$–given vector; $N\,w^T = (P^T,\, 0, Q^T)$, N–nonlinear operator. By virtue of the Theorem 2 we have

$$w = K^{-1}(h + Nw^T),\ \ \|w\|_t \le c_6\big(|||h|||_t + \sum_{r=1}^{N}(|P_r|_{\Omega_t}^{(\alpha)} + |Q_r|_{S_t}^{(1+\alpha)})\big),$$

here K^{-1}–inverse operator, the norms $\|w\|_t$ and $|||h|||_t$ are determined in formula (18). We derive the estimates

$$||K^{-1}(h^T + Nw)||_t \le c_6\,|||h|||_t + c_{12}\,\sqrt{t}\,||w||_t,\, ||K^{-1}(Nw - N\tilde{w})||_t \le c_{13}\,\sqrt{t}||w - \tilde{w}||_t.$$

Using these inequalities we apply the contractive mapping principle and obtain the unique solvability of the problem (38) in the closed ball $B_{T_0}(M) := \{w \mid v_r \in \overset{\circ}{C}{}_{x\ t}^{2+\alpha,1+\alpha/2}(\bar{\Omega}_{T_0}), r = 1, \ldots, N,\ \varphi_1 \in C_x^{2+\alpha}(\bar{\Omega}_T), ||w||_{T_0} \le M\}$ with the center in zero, $M = c_6\,|||h|||_T\,(1-q)^{-1}$, $q \in (0,1)$. Remembering the substitution (10) and estimates (14) and (9) for the vector h and the functions V_r we find the estimate (8) for v_r and φ_1. Then from the identities (21), (24), (25) with $p^T + P^T$ and $q^T + Q^T$ instead of p^T and q^T we obtain $\Delta\varphi_1 \in C_{x\ t}^{\alpha,\alpha/2}(\bar{\Omega}_T)$; $\varphi_1|_S, \partial_\nu\varphi_1|_S \in \overset{\circ}{C}{}_{x\ t}^{1+\alpha,(1+\alpha)/2}(S_T)$ and estimate (8) for them. Part 2 of theorem concerning the smoothness on t of potential φ_1 and it's derivatives is proved in the same way as in Theorem 2.•

4. Linear model problem..

We consider the problem (26)–(28) with unknown functions $u = (u_1, \ldots, u_N)$ satisfying zero initial data

$$\partial_t u^T - \big(E - \frac{m^T z}{\kappa}\big)\,A\,\Delta u^T = \frac{m^T}{\kappa}\partial_t F(x,t) + \big(E - \frac{m^T z}{\kappa}\big)\,f^T(x,t) \text{ in } \Omega_T, \tag{39}$$

$$zu^T = F(x,t) \text{ in } \bar{\Omega}_T, \tag{40}$$

$$\big(E - \frac{m^T z}{\kappa}\big)\,\big[E + \frac{1}{æ}\,G^T z A^{-1}\frac{m^T z}{\kappa}\big]\,A\,\partial_\nu u^T = \big(E - \frac{m^T z}{\kappa}\big)\,q^T(x,t) \text{ on } S_T. \tag{41}$$

Here the matrix A, vectors m, G, coefficients $\kappa = zm^T$, $æ = z\,A^{-1}(E - m^T z/\kappa)\,G^T$ are constant. We assume that matrix A has positive eigenvalues $\lambda_1, \ldots, \lambda_N$ and it may be reduced to the diagonal form.

Теорема 3. *Let $\kappa \ne 0$, $æ \ne 0$, $\alpha \in (0,1)$. Then for every functions $p_r \in \overset{\circ}{C}{}_{x\ t}^{\alpha,\,\alpha/2}(\bar{\Omega}_T)$, $q_r \in \overset{\circ}{C}{}_{x\ t}^{1+\alpha,(1+\alpha)/2}(S_T)$, $F \in \overset{\circ}{C}{}_{x\ t}^{2+\alpha,1+\alpha/2}(\bar{\Omega}_T)$ the problem (39)–(41) has unique solution $u_r \in \overset{\circ}{C}{}_{x\ t}^{2+\alpha,1+\alpha/2}(\bar{\Omega}_T), r = 1, \ldots, N$, and the estimate is valid*

$$\sum_{r=1}^{N} |u_r|_{\Omega_t}^{(2+\alpha)} \le c_{14}\big(\sum_{r=1}^{N}(|f_r|_{\Omega_t}^{(\alpha)} + |q_r|_{S_t}^{(1+\alpha)}) + |F|_{S_t}^{(2+\alpha)}\big),\ \ t \le T. \tag{42}$$

Let non-degenerate matrix Q transform the matrix A to the diagonal form

$$Q^{-1}\,A\,Q = \Lambda = diag(\lambda_1, \dots, \lambda_N).$$

The matrix $m^T\,z/\kappa$ is projector with rank equaled to 1 [6]. The same properties has similar to it matrix $Q^{-1}m^Tz/\kappa Q$. There exists orthogonal matrix P such that

$$P^TQ^{-1}\frac{m^Tz}{\kappa}QP = diag(\underbrace{0,\dots,0}_{N-1},1) := E_1 \ (P^T = P^{-1}),$$

here P^T-transposed matrix. We multiply both parts of the equation (39) and boundary condition (41) by the matrixes $P^TQ^{-1}(P^TQ^{-1}A^{-1}QP)^{-1} = P^TQ^{-1}\Lambda P$ and P^TQ^{-1} respectively and denote

$$v^T = P^T\,Q^{-1}Au^T, y = (y_1,\dots,y_N) = z\,A^{-1}\,Q\,P, l^T = P^T\,Q^{-1}\,G^T,$$

$$p^T(x,t) = \text{colon}\,(p_1,\dots,p_N) = P^T\,Q^{-1}\,f^T(x,t),\ h^T(x',t) = (h_1,\dots,h_N)^T =$$
$$= P^T\,Q^{-1}\,q^T(x',t),\ d^T = (d_1,\dots,d_N)^T = P^T\,Q^{-1}\,A\,m^T, \tag{43}$$

then we obtain the problem with unknown functions $v_1,\dots,v_N$

$$\partial_t v^T - P^T\Lambda P E_{N-1}\,\Delta v^T = \frac{1}{\kappa}d^T\partial_t F + P^T\Lambda P E_{N-1}\,p^T \text{ in } \Omega_T, \tag{44}$$

$$y\,v^T = F \text{ in } \bar\Omega_T, \tag{45}$$

$$E_{N-1}(E + \frac{1}{æ}l^TyE_1)\partial_\nu v^T = E_{N-1}h^T \text{ on } S_T, \tag{46}$$

here $E_{N-1} = E - E_1 = \text{diag}(\underbrace{1,\dots,1}_{N-1},0), æ = z\,A^{-1}(F - m^Tz/\kappa)\,G^T = y\,E_{N-1}\,l^T \neq 0.$

Let $P^T\Lambda P = \{b_{rs}\}_{1\le r,s\le N}$, $B = \{b_{rs}\}_{1\le r,s\le N-1}$, then the system of the equations (44) may be written as follows

$$\partial_t v'^T - B\,\Delta v'^T = \frac{1}{\kappa}d'^T\partial_t F + Bp'^T \text{ in } \Omega_T, \tag{47}$$

$$\partial_t v_N = \frac{1}{\kappa}\left(d_N - b'B^{-1}\,d'^T\right)\partial_t F + b'B^{-1}\,\partial_t v'^T \text{ in } \Omega_T. \tag{48}$$

The matrix $P^T\,\Lambda\,P$ is symmetric and positive definite, then B is also symmetric and positive definite matrix and the system (47) is parabolic. In (46) the matrix $1/æE_{N-1}l^Ty$ is projector of rank equaled to 1. Let $M = \{m_{rs}\}_{1\le r,s\le N}$ be the orthogonal matrix (not unique) such that
$M^T1/æE_{N-1}l^Ty\,M = \text{diag}(\underbrace{0,\dots,0}_{N-1},1) = E_1$ and $\det M_1 = \det\{m_{rs}\}_{1\le r,s\le N-1}$
$\neq 0$, then boundary conditions (46) may be written in the form

$$M_1\,M_1^T\partial_\nu v'^T = h' - \frac{1}{æ}\,l'^T\partial_\nu F \text{ on } R_T. \tag{49}$$

We have obtained parabolic boundary value problem (47), (49) which has unique solution v'. Integrating the equation (48) with respect to t we find v_N. Thus we have $v_r \in \overset{\circ}{C}{}_{x\ t}^{2+\alpha,1+\alpha/2}(\Omega_T)$, $r = 1, \dots, N$, and

$$\sum_{r=1}^{N} |v_r|_{\Omega_T}^{(2+\alpha)} \le c_{15}\big(\sum_{r=1}^{N-1}(|p_r|_{\Omega_T}^{(\alpha)} + |h_r|_{S_T}^{(1+\alpha)}) + |F|_{\Omega_T}^{(2+\alpha)}\big).$$

Remembering notations (43) we obtain estimate (42) and Theorem 3.●

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